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VARIATIONAL PRINCIPLES FOR NON-LINEAR ELASTOSTATICS IN EULERIAN COORDINATES*

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A stationarity principle is formulated for static problems in non-linear elasticity theory with the body boundary specified for the deformed state. Assuming material homogeneity, potential energy and complementary energy functionals are constructed for specified boundary conditions, together with Khu-Washizu and Tonti type functionals. The Hamilton-Ostrogradskii principle for dynamical problems in non-linear elasticity theory in Euler coordinates is considered in /1, 2/.

1. The system of equilibrium equations for a non-linearly elastic body has the form /3/

$$\operatorname{div} \mathbf{D} + \rho_0 \mathbf{b} = 0 \tag{1.1}$$

$$\mathbf{D} = \partial W_0/\partial \mathbf{C}, \ \mathbf{C} = \operatorname{grad} \mathbf{R}, \ \mathbf{R} = X_n \mathbf{i}_n \tag{1.2}$$

$$\operatorname{grad} = \mathbf{i}_k \partial/\partial x_k, \ \operatorname{div} \mathbf{D} \equiv \mathbf{i}_s \cdot \partial \mathbf{D}/\partial x_s$$

Here the x_k are Cartesian coordinates of points of the body in the undeformed (reference) configuration, i.e. Lagrangian coordinates, $\mathbf{i}_s$ are unit coordinate vectors, grad and div are gradient and divergence operators in the reference configuration, X_n are Cartesian coordinates of points of the body in the deformed state, i.e. Eulerian coordinates, $\mathbf{C}$ is the position gradient, $\mathbf{D}$ is the Piola stress tensor, $\mathbf{R}$ is the radius-vector of the point of the body in the deformed configuration, ρ_0 is the material density in the reference configuration, $\mathbf{b}$ is the external mass loading and W_0 is the specific potential energy of the strain per unit volume of the undeformed body.

System (1.1), (1.2) easily reduces to a system of equations for unknown X_k with independent variables x_s. This assumes that the region occupied by the elastic body in the reference configuration is known. Boundary conditions are formulated for a bounding surface σ whose equation is given in Lagrangian coordinates x_s. This elastostatic boundary-value problem in Lagrangian coordinates has a range of variational formulations derived in /4, 5/.

One could have a situation where the boundary of the elastic body Σ is specified in its deformed configuration, and the problem consists of finding the stress-strain state of the body for the given boundary conditions. In this case one should take the independent variables to be the Eulerian coordinates X_s and the unknown quantities to be the Lagrangian coordinates x_k.

In Eulerian coordinates the equilibrium equations may be written using the symmetric Cauchy stress tensor $\mathbf{T}$ /3/

$$\operatorname{Div} \mathbf{T} + \rho \mathbf{b} = 0, \quad \operatorname{Div} \mathbf{T} \equiv \mathbf{i}_k \cdot \partial \mathbf{T}/\partial X_k \tag{1.3}$$

where ρ is the material density in the deformed configuration and Div is the divergence operator in Eulerian coordinates.

Using the relation between the Cauchy and Piola tensors /3/

$$\mathbf{D} = (\det \mathbf{C})\, \mathbf{C}^{-T} \cdot \mathbf{T}, \quad \mathbf{C}^{-T} \equiv (\mathbf{C}^{-1})^T = (\mathbf{C}^T)^{-1} \tag{1.4}$$

together with the formulae

$$\partial (\det \mathbf{F}) / \partial \mathbf{F} = (\det \mathbf{F})\, \mathbf{F}^{-T}$$

$$\partial W/\partial \mathbf{C} = -\mathbf{F}^T \cdot (\partial W/\partial \mathbf{F}) \cdot \mathbf{F}^T$$

we obtain from (1.2) the defining relation for the Cauchy stress tensor:

$$\mathbf{T} = W\mathbf{E} - (\partial W/\partial \mathbf{F}) \cdot \mathbf{F}^T \tag{1.5}$$

$$\mathbf{F} = \mathbf{C}^{-1} = \operatorname{Grad} \mathbf{r} = \mathbf{i}_k \partial \mathbf{r}/\partial X_k, \ \mathbf{r} = x_s \mathbf{i}_s$$

$$W = (\det \mathbf{F})\, W_0 = W(\mathbf{F})$$

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Here $\mathbf{F}$ is the inverse position gradient, $\mathbf{r}$ is the radius-vector of the particle in the reference configuration, Grad is the gradient operator in Eulerian variables, W is the strain potential energy per unit volume of the deformed body and $\mathbf{E}$ is the unit tensor.

For a homogeneous body the specific energy W depends on the coordinates X_k only through the tensor $\mathbf{F}(X_k)$, i.e. W does not explicitly depend on X_k. Taking this into account we find an expression for $\operatorname{Div}\mathbf{T}$ with the help of (1.5):

$$\begin{gathered}\mathbf{i}_s\cdot\partial\,[W\mathbf{E}-(\partial W/\partial\mathbf{F})\cdot\mathbf{F}^T]\,/\,\partial X_s = \\ (\partial W/\partial F_{mn})\,(\partial F_{mn}/\partial X_s)\,\mathbf{i}_s - F_{kn}\partial\,(\partial W/\partial F_{sn})\,/\,\partial X_s\mathbf{i}_k - \\ (\partial W/\partial F_{sn})\,(\partial F_{kn}/\partial X_s)\,\mathbf{i}_k,\ \ F_{mn}=\mathbf{i}_m\cdot\mathbf{F}\cdot\mathbf{i}_n\end{gathered} \tag{1.6}$$

Because $F_{sn}=\partial x_n/\partial X_s$, we have $\partial F_{sn}/\partial X_k=\partial F_{kn}/\partial X_s$, from which

$$(\partial W/\partial F_{mn})\,(\partial F_{mn}/\partial X_s)\,\mathbf{i}_s=(\partial W/\partial F_{sn})\,(\partial F_{kn}/\partial X_s)\,\mathbf{i}_k \tag{1.7}$$

In accordance with (1.6) and (1.7) the equilibrium Eqs.(1.2) take the form

$$\operatorname{Div}\mathbf{K}-\rho\mathbf{b}\cdot\mathbf{F}^{-T}=0 \tag{1.8}$$

$$\mathbf{K}=\partial W/\partial\mathbf{F} \tag{1.9}$$

When there are no body forces the equilibrium Eqs.(1.8) and the governing Eqs.(1.9) are completely identical to Eqs.(1.1) and relations (1.2) except tht the reference and deformed configurations have swapped roles. Thus the non-symmetric tensor $\mathbf{K}$ can be considered to be an analogue of the Piola stress tensor for Eulerian coordinates.

This analogy does not extend to force boundary conditions, which have two equivalent forms in the notation of /3/:

$$\mathbf{n}\cdot\mathbf{D}=\mathbf{f}_0,\quad \mathbf{N}\cdot\mathbf{T}=\mathbf{f} \tag{1.10}$$

Here $\mathbf{n}$ and $\mathbf{N}$ are unit normal vectors in the reference and deformed frames respectively, $\mathbf{f}_0$ is the load across unit area of the surface σ, and $\mathbf{f}$ is the load crossing unit area of Σ. Eqs.(1.5) and (1.9) give the relation

$$\mathbf{K}=(W\mathbf{E}-\mathbf{T})\cdot\mathbf{F}^{-T} \tag{1.11}$$

which we use with (1.10) to obtain a formulation of the force boundary conditions in terms of the tensor $\mathbf{K}$

$$\mathbf{N}\cdot\mathbf{K}=(W\mathbf{N}-\mathbf{f})\cdot\mathbf{F}^{-T} \tag{1.12}$$

2. We assume that we know the boundary Σ of the elastic body in its deformed state and that it consists of three parts: $\Sigma=\Sigma_1\cup\Sigma_2\cup\Sigma_3$. On Σ_1 the displacement vector is specified, which is equivalent to specifying the vector $\mathbf{r}$ giving the position of the surface particle in the reference configuration. The external load is specified on Σ_3, i.e. (1.12) is satisfied. Points on the surface Σ_2 can slide smoothly along a specified solid surface without losing contact with it. The last condition reduces to the following restriction on variations $\delta\mathbf{r}$ on Σ_2:

$$\mathbf{n}\cdot\delta\mathbf{r}=0 \tag{2.1}$$

Using the transformation formula for an oriented surface element under strain,

$$\mathbf{n}\,d\sigma=(\det\mathbf{F})\,\mathbf{F}^{-1}\cdot\mathbf{N}\,d\Sigma$$

restriction (2.1) can be rewritten in the form

$$\mathbf{N}\cdot\mathbf{F}^{-T}\cdot\delta\mathbf{r}=0 \tag{2.2}$$

The zero-friction condition on the surface Σ_2,

$$\mathbf{N}\cdot\mathbf{T}\cdot\mathbf{G}=0,\quad \mathbf{G}=\mathbf{E}-\mathbf{N}\mathbf{N}$$

is, according to (1.11), equivalent to the requirement

$$\mathbf{N}\cdot\mathbf{K}\cdot\mathbf{F}^T\cdot\mathbf{G}=0 \tag{2.3}$$

We shall show that the equilibrium Eq.(1.8) and the boundary conditions (1.12) on Σ_3 and (2.3) on Σ_2 follow from the variational equation

$$\delta\int_V W\,dV=-\int_V\rho\mathbf{b}\cdot\mathbf{F}^{-T}\cdot\delta\mathbf{r}\,dV+\int_{\Sigma_3}(W\mathbf{N}-\mathbf{f})\cdot\mathbf{F}^{-T}\cdot\delta\mathbf{r}\,d\Sigma \tag{2.4}$$

The variations $\delta\mathbf{r}$ in (2.4) must satisfy the requirement $\delta\mathbf{r}=0$ on Σ_1 and condition (2.2) on Σ_2. From (1.9) we have

$$\delta\int_V W\,dV=\int_V \operatorname{tr}(\mathbf{K}\cdot\delta\mathbf{F}^T)\,dV=\int_\Sigma \mathbf{N}\cdot\mathbf{K}\cdot\delta\mathbf{r}\,d\Sigma-\int_V(\operatorname{Div}\mathbf{K})\cdot\delta\mathbf{r}\,dV= \tag{2.5}$$
$$\int_{\Sigma_2\cup\Sigma_3}\mathbf{N}\cdot\mathbf{K}\cdot\delta\mathbf{r}\,d\Sigma-\int_V(\operatorname{Div}\mathbf{K})\cdot\delta\mathbf{r}\,dV$$

Using (2.5), the restriction (2.2) on Σ_2, and also the equalities

$$\mathbf{N}\cdot\mathbf{K}\cdot\delta\mathbf{r}=\mathbf{N}\cdot\mathbf{K}\cdot\mathbf{F}^T\cdot\mathbf{G}\cdot\mathbf{F}^{-T}\cdot\delta\mathbf{r}+\mathbf{N}\cdot\mathbf{K}\cdot\mathbf{F}^T\cdot\mathbf{NN}\cdot\mathbf{F}^{-T}\cdot\delta\mathbf{r}$$
$$\mathbf{G}\cdot\mathbf{G}=\mathbf{G}$$

we transform the variational Eq.(2.4) to the form

$$\int(\operatorname{Div}\mathbf{K}-\rho\mathbf{b}\cdot\mathbf{F}^{-T})\cdot\delta\mathbf{r}\,dV-\int_{\Sigma_2}(\mathbf{N}\cdot\mathbf{K}\cdot\mathbf{F}^T\cdot\mathbf{G})\cdot(\mathbf{G}\cdot\mathbf{F}^{-T}\cdot\delta\mathbf{r})\,d\Sigma- \tag{2.6}$$
$$\int_{\Sigma_3}[\mathbf{N}\cdot\mathbf{K}-(W\mathbf{N}-\mathbf{f})\cdot\mathbf{F}^{-T}]\cdot\delta\mathbf{r}\,d\Sigma=0$$

Since the vector $\mathbf{G}\cdot\mathbf{F}^{-T}\cdot\delta\mathbf{r}$ on Σ_2 can take arbitrary values in the plane tangent to Σ_2, while in V and on Σ_3 the variations $\delta\mathbf{r}$ are arbitrary, the equilibrium Eqs.(1.8) and boundary conditions (1.12) and (2.3) follow from (2.6).

The expression on the right-hand side of (2.4) cannot, in general, be represented as the variation of some functional. Hence the variational Eq.(2.4) is not in general a variational principle, because it does not reduce to a stationarity condition for a functional. Furthermore, it is clear that even in the simplest case $\mathbf{b}=\text{const}$ the volume integral on the right of (2.4) will not be variational of a functional, while even when there is no boundary loading $(\mathbf{f}=0)$ the surface integral also fails to reduce to the variation of some functional.

In particular, it follows from this that even if one ignores body forces, the boundary-value problem of non-linear elastostatics in Eulerian coordinates with a free boundary component does not have a variational formulation, i.e. the problem is not equivalent to the stationarity of a functional.

Below we shall assume that there are no body forces and that the boundary Σ does not contain a component Σ_3 with a specified load. In that case the variational Eq.(2.4) turns into a variational principle for the stationarity of the potential energy

$$\delta\Pi[\mathbf{r}]=0,\quad \Pi=\int_V W\,dV$$

Relations (1.8) and (1.9) enable us to construct a Khu-Washizu type principle for the problem of non-linear elastostatics in Eulerian coordinates. In this principle the functions $\mathbf{r}$, $\mathbf{F}$ and $\mathbf{K}$ are varied independently, while the functional has the form

$$\Pi_1[\mathbf{r},\mathbf{F},\mathbf{K}]=\int_V[W(\mathbf{F})-\operatorname{tr}(\mathbf{K}^T\cdot(\mathbf{F}-\operatorname{Grad}\mathbf{r}))]\,dV-\int_{\Sigma_1}\mathbf{N}\cdot\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}^*)\,d\Sigma \tag{2.7}$$

Here $\mathbf{r}^*$ is the boundary value of the position vector in the reference configuration, specified on Σ_1. The main (stable) boundary condition for the functional (2.7) is the geometrical contact condition imposed on the vector $\mathbf{r}$ at Σ_2, reducing to restriction (2.2). The equilibrium Eqs.(1.8) with $\mathbf{b}=0$ serve as the Euler equations for the variational problem $\delta\Pi_1=0$, determining relations (1.9) and the geometric relation $\mathbf{F}=\operatorname{Grad}\mathbf{r}$. The boundary condition $\mathbf{r}=\mathbf{r}^*$ on Σ_1 and the zero friction condition (2.3) on Σ_2 fall out of the variational principle $\delta\Pi_1=0$, i.e. are nutural boundary conditions for the functional Π_1.

3. The system of equations for an elastic body in Eulerian coordinates can be transformed to a form not containing the function $\mathbf{r}(X_m)$ as an unknown. Elimination of this function leads to the following consistency condition:

$$\operatorname{Rot}\mathbf{F}=0,\ \operatorname{Rot}\mathbf{F}\equiv\mathbf{i}_k\times\partial\mathbf{F}/\partial X_k \tag{3.1}$$

When (3.1) is satisfied the vector field $\mathbf{r}(X_m)$ can be determined by quadratures in terms of the tensor field $\mathbf{F}(X_m)$.

If there are no body forces the equilibrium Eq.(1.8) can be satisfied by means of a tensor of stress functions Φ

$$\mathbf{K} = \mathrm{Rot}\,\mathbf{\Phi} \tag{3.2}$$

If with the help of the governing relations (1.9) we express the inverse position gradient $\mathbf{F}$ in terms of the stress tensor $\mathbf{K}$ and use (3.2), then the system of equations in Eulerian coordinates, describing the equilibrium of an elastic body, will consist of nine scalar consistency conditions (3.1) and contain the components of the tensor $\mathbf{\Phi}$ as unknowns.

As in the elastostatic problem in Lagrangian coordinates /4/, a system of equilibrium equations without displacements can be obtained from a Castigliano-type principle, for the formulation of which we introduce a complementary energy U - a function of the stress tensors $\mathbf{K}$, connected with the strain potential energy W by the Legendre transformation

$$U = \mathrm{tr}\,(\mathbf{K}\cdot\mathbf{F}^T) - W,\ \ \mathbf{F} = \partial U/\partial\mathbf{K} \tag{3.3}$$

It should be understood that here the complementary energy U is not the same as the complementary energy considered in /4/.

To construct the function $U(\mathbf{K})$ it is necessary to express the inverse position gradient $\mathbf{F}$ in terms of the stress tensor $\mathbf{K}$. We will consider this problem for the case of an isotropic material in which the strain potential energy W is a function of the invariants of a symmetric and positive-definite tensor $\mathbf{H} = \mathbf{A}\cdot\mathbf{F}^T$, where $\mathbf{A}$ is the orthogonal tensor associated with the strain /3/. The deformation measure $\mathbf{H}$ is associated with the Cauchy-Green finite strain tensor $\boldsymbol{\varepsilon}$ /3/ by the relation

$$\mathbf{H}^2 = (\mathbf{E} + 2\boldsymbol{\varepsilon})^{-1}$$

It follows from (1.9) that in an isotropic material the tensor $\mathbf{A}\cdot\mathbf{K}$ is symmetric and an isotropic function of the deformation measure $\mathbf{H}$:

$$\mathbf{A}\cdot\mathbf{K} = \varphi(\mathbf{H}) \tag{3.4}$$

Inverting (3.4) we obtain $\mathbf{H} = \psi(\mathbf{A}\cdot\mathbf{K})$.

For example, for a material with a potential energy function

$$W = {}^1/_2\lambda\,\mathrm{tr}^2\,(\mathbf{H} - \mathbf{E}) + \mu\,\mathrm{tr}\,(\mathbf{H} - \mathbf{E})^2,\ \ \lambda,\ \ \mu = \mathrm{const},$$

we have

$$\mathbf{A}\cdot\mathbf{K} = [\lambda\,(\mathrm{tr}\,\mathbf{H} - 3) - 2\mu]\,\mathbf{E} + 2\mu\mathbf{H}$$

$$\mathbf{H} - \mathbf{E} = (2\mu)^{-1}\left[\mathbf{A}\cdot\mathbf{K} - \frac{\nu}{1+\nu}\,\mathrm{tr}\,(\mathbf{A}\cdot\mathbf{K})\,\mathbf{E}\right],\quad \nu = \frac{\lambda}{2\,(\lambda+\mu)}$$

Because $\mathbf{F} = \mathbf{A}^T\cdot\mathbf{H}$, the problem of constructing the relation $\mathbf{F}(\mathbf{K})$ reduces to expressing the orthogonal tensor $\mathbf{A}$ in terms of the tensor $\mathbf{K}$:

$$\mathbf{F}(\mathbf{K}) = \mathbf{A}^T(\mathbf{K})\cdot\psi\,[\mathbf{A}(\mathbf{K})\cdot\mathbf{K}]$$

The expression $\mathbf{A}(\mathbf{K})$ is determined by solving the equation

$$\mathbf{A}\cdot\mathbf{K} = \mathbf{K}^T\cdot\mathbf{A}^T \tag{3.5}$$

expressing the symmetry of the tensor $\mathbf{A}\cdot\mathbf{K}$.

Eq.(3.5) is identical to an equation in /6/ which governs the dependence of the rotation tensor $\mathbf{A}$ on the Piola stress tensor. As was established in /6/, the solution of Eq.(3.5) is not unique and in general has four branches. Just as in /6/ it can be shown that only one of these branches corresponds to an angle of rotation for material line-elements that does not exceed 90°. Rejection of the remaining branches is accomplished by means of the inequality $\mathrm{tr}\,\mathbf{A} > 1$. Thus, if the rotation of material line-elements under the strain is not excessive, the representation of the inverse position gradient $\mathbf{F}$ in terms of the stress tensor $\mathbf{K}$ will be single-valued.

With the help of (3.2) we can express the complementary energy $U(\mathbf{K})$ in terms of the tensor stress function and consider the following Castigliano type functional:

$$\Pi_2[\mathbf{\Phi}] = \int_V U(\mathrm{Rot}\,\mathbf{\Phi})\,dV - \int_\Sigma \mathrm{tr}\,[\mathbf{N}\times\mathbf{\Phi}\cdot(\nabla\mathbf{r}^*)^T]\,d\Sigma \tag{3.6}$$

$$\nabla = \mathbf{G}\cdot\mathrm{Grad}$$

Here $\Sigma = \Sigma_1$, while ∇ denotes the two-dimensional gradient operator on the surface /7/. The variation of the functional (3.6) has the form

$$\delta\Pi_2 = \int_V \mathrm{tr}\,(\delta\mathbf{\Phi}^T\cdot\mathrm{Rot}\,\mathbf{F})\,dV + \int_\Sigma \mathrm{tr}\,[(\mathbf{N}\times\delta\mathbf{\Phi})\cdot(\mathbf{G}\cdot\mathbf{F} - \nabla\mathbf{r}^*)^T]\,d\Sigma \tag{3.7}$$

Expression (3.7) shows that demanding the stationarity of the functional Π_2 is equivalent to the consistency Eqs.(3.1) and the boundary conditions

$$\mathbf{G}\cdot\mathbf{F} = \nabla\mathbf{r}^* \qquad (3.8)$$

If the surface Σ is simply connected, then condition (3.8) is equivalent to the conditions $\mathbf{r} = \mathbf{r}^*$. Indeed, differentiating the latter, we obtain (3.8), while integrating (3.8) we find $\mathbf{r} = \mathbf{r}^* + \mathbf{d}$, where $\mathbf{d}$ is an arbitrary constant vector. Because the imposition of a translation does not alter the stress-strain state of the body, conditions (3.8) are equivalent to specifying displacements at the boundary of the domain.

For an isotropic homogeneous material one can formulate a weak complementary energy principle similar to the variational principle suggested in /8, 9/. In this case, along with the possible static stresses the orthogonal tensor field $\mathbf{A}$ is independently varied, and the following expression is used for the complementary energy:

$$U(\mathbf{P}) = \operatorname{tr}(\mathbf{P}\cdot\mathbf{H}) - W, \quad \mathbf{P} = {}^1/_2(\mathbf{A}\cdot\mathbf{K} + \mathbf{K}^T\cdot\mathbf{A}^T) = \partial W/\partial\mathbf{H} \qquad (3.9)$$

From (3.9) we obtain

$$\delta U = \operatorname{tr}(\mathbf{H}\cdot\delta\mathbf{P}) = \operatorname{tr}(\mathbf{F}^T\cdot\delta\mathbf{K}) + \operatorname{tr}(\mathbf{K}\cdot\mathbf{F}^T\cdot\mathbf{A}^T\cdot\delta\mathbf{A}) \qquad (3.10)$$

In Eulerian coordinates the functional for the weak complementary energy principle has the form

$$\Pi_3[\mathbf{\Phi}, \mathbf{A}] = \int\limits_V U(\operatorname{Rot}\mathbf{\Phi}, \mathbf{A})\,dV - \int\limits_\Sigma \operatorname{tr}[(\mathbf{N}\times\mathbf{\Phi})\cdot(\nabla\mathbf{r}^*)^T]\,d\Sigma \qquad (3.11)$$

By virtue of (3.10) and the properties of the antisymmetric tensor $\mathbf{A}^T\cdot\delta\mathbf{A}$ the stationarity conditions for functionals (3.11) consist of stationarity conditions for the functional Π_2 and the equations $\mathbf{K}\cdot\mathbf{F}^T = \mathbf{F}\cdot\mathbf{K}^T$, which together with (1.11) express the symmetry property of the Cauchy stress tensor $\mathbf{T}$.

In conclusion we give an expression for the functional of a Tonti type variational principle

$$\Pi_4[\mathbf{F}, \mathbf{\Phi}] = \int\limits_V [\operatorname{tr}(\operatorname{Rot}\mathbf{\Phi}\cdot\mathbf{F}^T) - W(\mathbf{F})]\,dV - \int\limits_\Sigma \operatorname{tr}[(\mathbf{N}\times\mathbf{\Phi})\cdot(\nabla\mathbf{r}^*)^T]\,d\Sigma$$

The stationarity conditions for the functional Π_4 are the equations

$$\operatorname{Rot}\mathbf{\Phi} = \partial W/\partial\mathbf{F}, \quad \operatorname{Rot}\mathbf{F} = 0$$

and boundary conditions (3.8) on Σ.

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